

# A REFLECTION PRINCIPLE FOR REAL-ANALYTIC SUBMANIFOLDS OF COMPLEX SPACES

BERNHARD LAMEL

ABSTRACT. We give an invariant nondegeneracy condition for CR-maps between generic submanifolds in different dimensions and use it to prove a reflection principle for these maps.

## 1. AN INVARIANT CONDITION FOR CR-MAPS

Assume that  $M \subset \mathbb{C}^N$  and  $M' \subset \mathbb{C}^{N'}$  are generic real-analytic submanifolds of  $\mathbb{C}^N$  and  $\mathbb{C}^{N'}$ , respectively. We will consider CR-maps  $H : M \rightarrow M'$ , that is, maps which push complex tangent vectors to  $M$  to complex tangent vectors to  $M'$ , ie,  $H_*T_p^c M \subset T_{H(p)}^c M'$ . For our purposes, this means that after choosing coordinates  $(Z'_1, \dots, Z'_{N'}) \in \mathbb{C}^{N'}$  the components to  $H$  in this coordinate system are CR-functions on  $M$  (For details concerning CR-maps and the last statement, the reader is encouraged to see [1], §2.3.). We want to give a condition on the map  $H$  which guarantees that it is the restriction of a holomorphic map from  $\mathbb{C}^N$  into  $\mathbb{C}^{N'}$ . Our nondegeneracy condition involves taking derivatives of the complex gradients to  $M'$  pulled back to  $M$  via  $H$  with respect to the CR-vector fields tangent to  $M$ . For simplicity, assume that  $0 \in M$ ,  $0 \in M'$  and  $H(0) = 0$  (in the examples, this will not always be the case). Let  $\rho' = (\rho'_1, \dots, \rho'_{d'})$  be a local defining function for  $M'$  near  $0 \in \mathbb{C}^{N'}$ . We write  $\mathcal{V}(M)$  for the CR-bundle of  $M$ .

**Definition 1.** Assume that  $H$  is a CR-mapping of  $M$  into  $M'$  of class  $C^{k_0}$  and  $H(0) = 0$ . Define complex linear subspaces  $E_k(0) \subset \mathbb{C}^{N'}$  by

$$(1) \quad E_k(0) = \text{span}_{\mathbb{C}}\{L_1 \cdots L_j \rho'_{sZ'} \left( H(Z), \overline{H(Z)} \right) \Big|_{Z=0} : L_r \in \Gamma(M, \mathcal{V}(M)) \\ \text{for } 1 \leq r \leq j, 0 \leq j \leq k, 1 \leq l \leq d'\}.$$

We say that  $H$  is  $k_0$ -nondegenerate if  $E_{k_0}(0) = \mathbb{C}^{N'}$  and  $E_k(0) \neq \mathbb{C}^{N'}$  for  $k < k_0$ . Here we have used the notation  $\rho'_{sZ'} = (\rho'_{sZ'_1}, \dots, \rho'_{sZ'_{N'}})$  for the complex gradient.

We will show that Definition 1 is invariant under biholomorphic changes of coordinates in Lemma 3.

**Remark 1.** If there exists a  $k_0$ -nondegenerate map into the manifold  $M'$ , then automatically  $M'$  is finitely nondegenerate. Here we say that a real submanifold  $M$  is finitely nondegenerate (or more specifically  $k_0$ -nondegenerate) if the identity mapping of  $M$  is  $k_0$ -nondegenerate in the sense of Definition 1. For the original definition of finite nondegeneracy, see [1], §11.1.. In fact, if there is a  $k_0$ -nondegenerate map  $H$  into  $M'$ , then  $M'$  will be  $\ell_0$ -nondegenerate with  $k_0 \geq \ell_0$ . The reader can

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check this by writing out the condition in Definition 1 in coordinates and using the chain rule.

Let  $\Gamma$  be an open, convex cone in  $\mathbb{R}^d$ . By a wedge  $\hat{W}_\Gamma \subset \mathbb{C}^N$  with edge  $M$  we mean a set of the form  $\hat{W}_\Gamma = \{Z \in U : \rho(Z, \bar{Z}) \in \Gamma\}$  where  $\rho$  is a defining function for  $M$  and  $U \subset \mathbb{C}^N$  is an open neighbourhood of the origin. We can now state our result.

**Theorem 2.** *Let  $M \subset \mathbb{C}^N$ ,  $M' \subset \mathbb{C}^{N'}$  be generic real-analytic submanifolds,  $0 \in M$ ,  $0 \in M'$ . Assume that  $H : M \rightarrow M'$  is a CR-map with  $H(0) = 0$  which is  $k_0$ -nondegenerate at 0 and extends continuously to a holomorphic function (called again  $H$ ) in a wedge  $\hat{W}_\Gamma$  with edge  $M$ . Then  $H$  extends to a holomorphic map  $H : \mathbb{C}^N \rightarrow \mathbb{C}^{N'}$  in a full neighbourhood of  $0 \in \mathbb{C}^N$ .*

Let us note that if we assume  $M$  to be of finite type, then by a result of Tumanov [12] every CR-map  $H$  will automatically extend to a wedge. Also, the hypotheses that  $H$  extends continuously and that  $H$  is  $C^{k_0}$  on  $M$  implies that actually  $H$  extends as a  $C^{k_0}$  function up to the edge  $M$  of  $\hat{W}_\Gamma$  (see e.g. [1], §7.5.).

Note that if  $M \subset \mathbb{C}^N$ ,  $M' \subset \mathbb{C}^{N'}$  are of equal codimension and both  $\ell_0$ -nondegenerate, then every CR-diffeomorphism of class  $\ell_0$  is  $\ell_0$ -nondegenerate. Let us now illustrate Definition 1 by giving some examples.

**Example 1.** Let  $M = \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^4\}$ ,  $M' = \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2\}$ . Then the map  $H(z, w) = (z^2, w)$  is 2-nondegenerate.

We will now give an example where  $M$  has codimension 1 and  $M'$  has codimension 2.

**Example 2.** Consider the manifolds  $M = \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2\}$ ,  $M' = \{(z', w'_1, w'_2) : \operatorname{Im} w'_1 = \operatorname{Im} w'_2 = |z'|^2\}$ . The map  $H : \mathbb{C}^2 \rightarrow \mathbb{C}^3$ ,  $H(z, w) = (z, w, w)$  is 1-nondegenerate.

**Example 3.** As the source manifold consider the ball in  $\mathbb{C}^2$ ,  $M = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ . There are (up to automorphisms) exactly three nonlinear maps from  $M$  into the 3-ball  $M' = \{(z'_1, z'_2, z'_3) \in \mathbb{C}^3 : |z'_1|^2 + |z'_2|^2 + |z'_3|^2 = 1\}$ , as Faran [5] showed:  $H_1(z_1, z_2) = (z_1, z_1 z_2, z_2^2)$ ,  $H_2(z_1, z_2) = (z_1^2, \sqrt{2} z_1 z_2, z_2^2)$ , and  $H_3(z_1, z_2) = (z_1^3, \sqrt{3} z_1 z_2, z_2^3)$ . These are 2, 2, and 3-nondegenerate, respectively, at the point  $(1, 0, 0)$ . However, if we consider maps from an  $n$ -ball into an  $n+1$ -ball which are at least  $C^3$  for  $n \geq 3$ , a result due to Webster [13] tells us that these are all linear and hence never nondegenerate.

We would like to note that variants of Definition 1 have appeared in the literature before, for example in the case of pseudoconvex hypersurfaces ([4], [6]) but the invariant formulation given here is new. Another similar condition can be found in [9]. In this paper Han studies the reflection principle for Levi-nondegenerate hypersurfaces. He requires that after putting the target hypersurface in suitable coordinates  $(z', w')$  and writing  $H = (f, g)$  in these coordinates, that the derivatives  $L^\alpha \bar{f}(0)$  where  $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$  and  $L_1, \dots, L_n$  is a basis for the CR-vector fields tangent to  $M$  span  $\mathbb{C}^{n'}$  (where  $N' = n' + 1$ ). The problem with this condition is that it is not invariant under biholomorphic changes of coordinates, and Theorem 1.1 in [9] is incorrect, as the following example which is due to D. Zaitsev shows.

**Example 4.** Consider the hypersurface  $M' \subset \mathbb{C}^3$  given by  $\operatorname{Im} w = |z_1|^2 - |z_2|^2$ . After the change of coordinates  $z_1 = \zeta_1 + \zeta_2 - \zeta_2^2$ ,  $z_2 = \zeta_2$ ,  $w = \tau$ ,  $M$  takes the form  $\operatorname{Im} \tau = |\zeta_1 + \zeta_2 - \zeta_2^2|^2 - |\zeta_2|^2$ . Let  $\phi$  be a CR-function on  $M = \{(z, w) : \operatorname{Im} w = |z|^2\}$  and consider the map  $H(z, w) = (\phi^2(z, w), \phi(z, w), 0)$ . Then  $f = (\phi^2, \phi)$  and if we choose  $\phi$  such that  $\phi_z(0) \neq 0$ , then the derivatives  $L\bar{f}(0)$  and  $L^2\bar{f}(0)$  will span  $\mathbb{C}^2$ . Choosing for  $\phi$  any CR-function which does not extend (such exist since  $M$  is strongly pseudoconvex) gives a counterexample to Theorem 1.1 in [9].

We would now like to mention some of the reflection principles which have been obtained for manifolds in complex spaces of different dimensions. In the case of pseudoconvex hypersurfaces, reflection principles were obtained in [7] and [10]. A reflection principle for balls has been obtained in [2]. Also, the reader is referred to the survey article [8] for some other special cases which have been settled.

We will conclude this section by proving that Definition 1 is independent of the choice of coordinates. The definition is independent of the choice of defining function, as the reader can easily prove by induction on  $k$ . The definition is also independent of the choice of coordinates  $Z \in \mathbb{C}^N$ . It only remains to be shown that it is independent of changes of holomorphic coordinates in the target space.

**Lemma 3.** *Let  $M \subset \mathbb{C}^N$ ,  $M' \subset \mathbb{C}^{N'}$ ,  $H$  be as in Definition 1. Define  $E_k(0)$  by (1). If we change coordinates by  $\tilde{Z}' = F(Z')$  in  $\mathbb{C}^{N'}$ , and denote the corresponding spaces defined by (1) in the new coordinates by  $\tilde{E}_k(0)$ , then*

$$(2) \quad \tilde{E}_k(0) = E_k(0) \left( \frac{\partial F}{\partial Z'}(0) \right)^{-1},$$

where  $E_k(0)$  is considered as a space of row vectors.

*Proof.* Suppose that  $F : \mathbb{C}^{N'} \rightarrow \mathbb{C}^{N'}$  is a local biholomorphic change of coordinates with  $F(0) = 0$ . Since Definition 1 is independent of the choice of defining function we can choose  $\tilde{\rho} = \rho' \circ F^{-1}$  as a defining function for  $F(M)$ . By the chain rule we have that

$$\tilde{\rho}_{j\tilde{Z}'} = (\rho'_j \circ F^{-1})_{\tilde{Z}'} = (\rho'_{jZ'} \circ F^{-1}) \left( \frac{\partial F^{-1}}{\partial \tilde{Z}'} \right).$$

Now the only crucial point is that the matrix on the right hand side has holomorphic entries. Hence, if we compose with  $H$ , this matrix will be annihilated by the CR-vector fields on  $M$ . Applying the vector fields  $L_1, \dots, L_r$  as in (1) to this equation we conclude that

$$(3) \quad L_1 \cdots L_r \tilde{\rho}_{j\tilde{Z}'}(F \circ H(Z), \overline{F \circ H(Z)}) \\ = L_1 \cdots L_r \rho'_{jZ'}(H(Z), \overline{H(Z)}) \left( \frac{\partial F^{-1}}{\partial \tilde{Z}'}(F(H(Z))) \right).$$

Evaluating this identity at 0, we obtain (2).  $\square$

## 2. PRELIMINARIES

Before we give the proof of Theorem 2, we want to give the technical details on which the proof is based. We will be making use of the Edge-of-the-Wedge Theorem. The technique used here is basically the same as in [3]. We start by fixing a defining equation for  $M$ . Assume that  $\operatorname{codim} M = d$ . Then it is possible

to choose coordinates  $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d = \mathbb{C}^N$  near 0 and a real-analytic function  $\varphi : \mathbb{C}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ , defined near 0, such that  $M$  is given by

$$(4) \quad \operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

and  $\varphi(0) = 0$ ,  $d\varphi(0) = 0$ ,  $\varphi(z, 0, s) = \varphi(0, \bar{z}, s) = 0$ . Define the function  $\Psi : \mathbb{C}^n \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}^N$ ,

$$\Psi(z, \bar{z}, s, t) = (z, s + it + i\varphi(z, \bar{z}, s + it)).$$

This is a real-analytic diffeomorphism, but *not* holomorphic. It flattens out  $M$  in the sense that near the origin  $\Psi^{-1}(M) = \mathbb{C}^n \times \mathbb{R}^d \times \{0\}$ . With respect to  $s + it$ , that is, for  $z$  fixed,  $\Psi$  is actually holomorphic. We shall equip  $\mathbb{C}^n \times \mathbb{R}^d$  with the CR-structure transported by  $\Psi$  from  $M$ . Let  $\Gamma$  be an open, convex cone in  $\mathbb{R}^d$  (all cones will be assumed to be open and convex). By a wedge  $W_\Gamma$  we mean a set of the form

$$W_\Gamma = \{(z, s, t) \in \mathbb{C}^n \times \mathbb{R}^d \times \mathbb{R}^d : t \in \Gamma\}.$$

We will also call the image under  $\Psi$  of  $W_\Gamma$  a wedge, but to avoid confusion we will write  $\tilde{W}_\Gamma$  for it. We will also write  $W_\Gamma^\epsilon = \{(z, s, t) \in W_\Gamma : |z| < \epsilon, |s| < \epsilon, |t| < \epsilon\}$  and  $\tilde{W}_\Gamma^\epsilon$  for the image under  $\Psi$  of  $W_\Gamma^\epsilon$ . The apparent difference with the definition of the wedge  $\hat{W}_\Gamma$  given before the statement of Theorem 2 is reconciled by the observation that given any cone  $\Gamma'$  whose closure is contained in  $\Gamma$ , then in the sense of germs of sets,  $\hat{W}_{\Gamma'} \subset \tilde{W}_\Gamma$  and  $\tilde{W}_{\Gamma'} \subset \hat{W}_\Gamma$  (see the end of Section 2 of [3]).

By saying that a CR-function  $h$  on  $M$  extends holomorphically to  $W_\Gamma^\epsilon$  we shall mean that there is a holomorphic function  $f$  on  $\tilde{W}_\Gamma^\epsilon$ , continuous up to  $M$ , such that  $h(z, \bar{z}, s) = f(\Psi(z, \bar{z}, s, 0))$ , and likewise we will say that  $h$  extends holomorphically to a full neighbourhood of the origin if there is a holomorphic function  $f$  defined in a full neighbourhood of the origin such that this equality holds. To say that  $h$  extends holomorphically to a full neighbourhood of the origin is the same as having an  $\epsilon > 0$  such that for each  $z$  with  $|z| < \epsilon$  the function  $s \mapsto h(z, \bar{z}, s)$  extends holomorphically in  $s + it$  for  $|s| < \epsilon$ ,  $|t| < \epsilon$  such that the function  $h(z, \bar{z}, s, t)$  is a bounded, measurable function in all of its variables for  $|z| < \epsilon$ ,  $|s| < \epsilon$ ,  $|t| < \epsilon$  (for details, see e.g. [1], §1.7.).

The Edge-of-the-Wedge theorem ensures that this will be the case if  $h$  extends holomorphically in  $s + it$  for each fixed  $z$  to  $W_\Gamma^\epsilon$  and  $W_{-\Gamma}^\epsilon$  (for details, see Section 2 of [3]) in a way such that  $h$  is a continuous function of all its variables in  $W_\Gamma^\epsilon \cup W_{-\Gamma}^\epsilon \cup (M \cap \{|z| < \epsilon, |s| < \epsilon\})$ . Actually, this assumption might seem unnecessarily strong to the reader; in fact, it is enough to assume that the extension has boundary value in the sense of distributions  $h$ , but a regularity theorem implies that if  $h$  is continuous on  $M$ , then this extension is actually continuous in the sense above (see e.g. [1], §7.3. and [11]). Since we shall only deal with functions which are at least continuous on  $M$ , we shall assume continuity of the extension a priori, and shall not go into further detail.

For the following arguments, we will fix  $\Gamma$  and say that  $h$  *extends up* differentiably (respectively continuously) if it extends holomorphically to  $W_\Gamma^\epsilon$  in a way such that the extended function is a  $C^1$  (respectively continuous) function of all of its variables in  $\tilde{W}_\Gamma^\epsilon = W_\Gamma^\epsilon \cup M$  (where  $M$  is shrunk appropriately) and that  $h$  *extends down* if it extends holomorphically to  $\tilde{W}_{-\Gamma}^\epsilon$  in a way such that the extended function is a  $C^1$  (respectively continuous) function of all of its variables (this is the terminology used in [1], §9.2., in the case of hypersurfaces).

The main point can be phrased as follows:

*If  $h$  extends up continuously then  $\bar{h}$  extends down continuously.*

In fact, the extension of  $\bar{h}$  is just given by  $\bar{h}(z, \bar{z}, s, -t) = \overline{h(z, \bar{z}, s, t)}$ . This will be used later on for the components of the CR-mapping  $H$ . We also need to know how  $Lh$  behaves for vector fields  $L$  tangent to  $M$  whose coefficients extend up continuously. Here the result is that if the coefficients of  $L$  extend up continuously and  $h$  extends up differentiably, then  $Lh$  extends up continuously. Similarly, if the coefficients of  $L$  extend up smoothly, the regularity of the extension is dropped by 1 (that is, if  $h$  extends up in a  $C^k$  manner,  $Lh$  does so in a  $C^{k-1}$  manner). We summarize the discussion:

**Lemma 4.** *Let  $h$  be a CR-function on  $M$ . If  $h$  extends up continuously (respectively differentiably), then  $\bar{h}$  extends down continuously (respectively differentiably). If  $L$  is a vector field tangent to  $M$  whose coefficients extend up smoothly and  $h$  extends up of order  $C^k$  then  $Lh$  extends up of order  $C^{k-1}$ . If  $h$  extends up and down continuously, then  $h$  extends holomorphically to a full neighbourhood of the origin.*

The next lemma is used in order to actually calculate with Definition 1. Its proof is an easy induction on  $k$  which is left to the reader.

**Lemma 5.** *Let  $L_1, \dots, L_n$  be a local basis near 0 of  $\Gamma(M, \mathcal{V}(M))$ . For every multiindex  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  define  $L^\alpha = L_1^{\alpha_1} \dots L_n^{\alpha_n}$ . Then*

$$(5) \quad E_k(0) = \text{span}_{\mathbb{C}}\{L^\alpha \rho'_{lZ'}(H(Z), \overline{H(\bar{Z})})|_{Z=0} : 1 \leq l \leq d', |\alpha| \leq k\},$$

where the  $E_k(0)$  are defined by (1).

### 3. PROOF OF THE REFLECTION PRINCIPLE

Choose coordinates  $(Z'_1, \dots, Z'_{N'}) \in \mathbb{C}^{N'}$  and a real-analytic defining function  $\rho'$  for  $M'$  defined in some neighbourhood of 0. In these coordinates,  $H = (H_1, \dots, H_{N'})$  with each  $H_j$  a CR-function on  $M$ . We will think of  $M$  as  $\mathbb{C}^n \times \mathbb{R}^d$  as explained in Section 2. A local basis of the CR-vector fields (for which we will use the vector notation and write  $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ ) is then given in matrix notation by (with  $\varphi$  as in (4); we also refer the reader to [1], §1.6.)

$$(6) \quad \Lambda = \frac{\partial}{\partial \bar{z}} - i(\varphi_{\bar{z}})^T (I + i\varphi_s)^{-1} \frac{\partial}{\partial s}.$$

The coefficients of all of these vector fields clearly extend up and down (smoothly), since  $\varphi$  is real-analytic. Since  $H$  maps  $M$  into  $M'$ ,  $\rho'(H(Z), \overline{H(\bar{Z})}) = 0$  for  $Z \in M$ . Applying  $\Lambda_1, \dots, \Lambda_n$  repeatedly and using the chain rule we see that for every multiindex  $\alpha \in \mathbb{N}^n$ ,  $|\alpha| \leq k_0$  (as in Lemma 5,  $\Lambda^\alpha = \Lambda^{\alpha_1} \dots \Lambda^{\alpha_n}$ ) and for every  $l$ ,  $1 \leq l \leq d'$ ,

$$(7) \quad 0 = \Lambda^\alpha \rho'_l(H(Z), \overline{H(\bar{Z})}) = \Phi_{l\alpha}(H(Z), \overline{H(\bar{Z})}, (\Lambda^\beta \overline{H(\bar{Z})})_{1 \leq |\beta| \leq k_0}), \quad Z \in M,$$

where  $\Phi_{l\alpha}$  is a real-analytic function defined and convergent on a neighbourhood of  $\{0\} \times \{0\} \times \mathbb{C}^{K(k_0)} \subset \mathbb{C}^{N'} \times \mathbb{C}^{N'} \times \mathbb{C}^{K(k_0)}$ ,  $K(k_0)$  being the cardinality of the set  $\{\beta \in \mathbb{N}^n : 1 \leq |\beta| \leq k_0\}$ . By our assumption and Lemma 5 we can choose  $N'$  multiindices  $\alpha^1, \dots, \alpha^{N'}$ ,  $0 \leq |\alpha^j| \leq k_0$  and integers  $l^1, \dots, l^{N'}$ ,  $1 \leq l^j \leq d'$ , such that

$$(8) \quad \text{span}_{\mathbb{C}}\{\Lambda^{\alpha^j} \rho'_{l^j Z'}(H(Z), \overline{H(\bar{Z})})|_{Z=0} : 1 \leq j \leq N'\} = \mathbb{C}^{N'}.$$

We consider the system of equations

$$(9) \quad \Phi_{l^j \alpha^j}(X_1, \dots, X_{N'}, Y_1, \dots, Y_{N'}, W) = 0, \quad 1 \leq j \leq N',$$

where  $W \in \mathbb{C}^{K(k_0)}$ . We claim that (9) admits a (unique) real-analytic solution in  $(X_1, \dots, X_{N'})$  in a neighbourhood of the point  $(0, 0, (\Lambda^\beta \bar{H}(0))_{1 \leq |\beta| \leq k_0})$ . In fact, if we compute the Jacobian of this system with respect to  $X_1, \dots, X_{N'}$  it is of full rank at this point because of (8). So we can invoke the implicit function theorem to conclude that there are real-analytic functions  $\Upsilon_1, \dots, \Upsilon_{N'}$ , convergent on a neighbourhood  $U$  of  $(0, (\Lambda^\beta \bar{H}(0))_{1 \leq |\beta| \leq k_0})$ , such that the unique solution of (9) in  $U$  is given by

$$(10) \quad X_j = \Upsilon_j(Y_1, \dots, Y_{N'}, W), \quad 1 \leq j \leq N'.$$

Recalling (7) we conclude that

$$(11) \quad H_j(Z) = \Upsilon_j(\overline{H_1(Z)}, \dots, \overline{H_{N'}(Z)}, (\Lambda^\beta \overline{H(Z)})_{1 \leq |\beta| \leq k_0}).$$

Now the proof is finished by using Lemma 4: Each  $H_j$  is assumed to extend up; so each  $\bar{H}_j$  extends down (of order  $C^{k_0}$ , by a regularity theorem, see Theorem 7.5.1. in [1]). Hence the whole right hand side of (11) extends down continuously (after choosing  $\epsilon$  small enough). This shows that each  $H_j$  extends down continuously. Since each  $H_j$  also extends up, Lemma 4 implies that each  $H_j$  extends holomorphically to a full neighbourhood of the origin. The theorem is proved.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SAN DIEGO, LA JOLLA, CA 92092-0112

*E-mail address*: blamel@euclid.ucsd.edu